Comment on "Spin-glass attractor on tridimensional hierarchical lattices in the presence of an external magnetic field"

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The effect of random or uniform magnetic fields on spin glasses on some $d=3$ hierarchical lattices has been studied, using renormalization-group theory, by Salmon and Nobre. In this work, the recursion relation for the local magnetic field is incorrect. It erroneously weakens the amplitude of the renormalization contribution to the magnetic field by a factor of $1/q_\mu$, where q_μ is the site coordination number.

I. INTRODUCTION

Hierarchical lattices, introduced in Refs. [1–3], provide exact renormalization-group solutions to diverse complex problems, including magnetic systems with quenched randomness, as seen in recent works [4]. The distinctive advantage of a system on a hierarchical lattice is that it is a physically realizable system [1]. Thus, it is solved exactly in its uniquely defined way, yielding the entire thermodynamics of the system [5,6], with no ambiguity, no *ad hoc* procedure, no "recipe" needed, in contrast to uncontrolled position-space renormalization-group approximations.

The Ising spin-glass system was studied under random fields on a $d=3$ hierarchical lattice [7]. The construction (and the renormalization-group solution) of this hierarchical lattice employed a rescaling factor of $b=3$ that preserves the ferromagnetic-antiferromagnetic symmetry of loose-packed lattices. It was found that the finite-temperature spin-glass phase that occurs at zero field disappears, being replaced by the disordered phase, with the application of even an infinitesimal random field. By a local gauge transformation argument, it was deduced that an infinitesimal uniform field has the same effect of eliminating the finite-temperature spin-glass phase in favor of the disordered phase.

Recently, the $d=3$ Ising spin-glass system has been studied on some hierarchical lattices with $b=2$ [8]. The systems in [7,8] are physically differently constructed systems and can therefore not be compared in the detail. However, qualitatively speaking, contrary to the previous work [7], [8] has found that the finite-temperature spin-glass phase persists under finite strengths of random or uniform fields. This result of [8] is simply due to an error in the renormalization-group recursion relation, strongly underestimating the effect of the magnetic field and not achieving an exact renormalization-group transformation, as explained below. The magnetic recursion relation of [8] fails to be correct even for the $d=1$ Ising model, also as explained below.

II. ERROR IN THE RENORMALIZATION CONTRIBUTION TO THE LOCAL MAGNETIC FIELD

In Ref. [8], on the left-hand side of the recursion relation in Eq. (13) for the renormalized local magnetic field, the second term gives the renormalization contribution to the local field at site μ and there should obviously be such a term for each renormalized bond connected to site μ, namely, the correct equation is

$$H'_\mu = H_\mu + \sum_\nu \left[\frac{1}{4}\ln\left(\frac{\mathcal{Z}^{\mu\nu}_{++}\mathcal{Z}^{\mu\nu}_{+-}}{\mathcal{Z}^{\mu\nu}_{--}\mathcal{Z}^{\mu\nu}_{-+}} \right) \right], \qquad (1)$$

where the sum is over all neighbors ν of the site μ in the renormalized system. The sum is missing in Ref. [8]. (This sum is also clearly indicated, for example, in the caption of Fig. 1 in Cao and Machta [9].) The notation and the definitions of Ref. [8] are used throughout this Comment.

The same applies for Eq. (14) in [8]. This error is easily avoided when the fields are counted with the bonds, as in [7]. This is because the initial Hamiltonian of Eq. (1) in Ref. [8],

$$-\beta\mathcal{H} = \sum_{\langle ij \rangle} J_{ij}s_i s_j + \sum_i H_i s_i, \qquad (2)$$

becomes, after the first renormalization-group transformation,

$$-\beta\mathcal{H}' = \sum_{\langle ij \rangle} J'_{ij}s_i s_j + \sum_{\langle ij \rangle} (\widetilde{H}_{ij}s_i + \widetilde{H}_{ji}s_j) + \sum_i H_i s_i, \qquad (3)$$

where the second term on the right-hand side reflects the second term in the above Eq. (1) in the current Comment,

$$\widetilde{H}_{\mu\nu} = \frac{1}{4}\ln\left(\frac{\mathcal{Z}^{\mu\nu}_{++}\mathcal{Z}^{\mu\nu}_{+-}}{\mathcal{Z}^{\mu\nu}_{--}\mathcal{Z}^{\mu\nu}_{-+}} \right), \qquad (4)$$

and, in the third term on the right-hand side of Eq. (3), H_i is the initial magnetic field appearing in Eq. (2) above. After the (n)th renormalization-group transformation,

FIG. 1. The repeated imbedding of the graph as shown in this figure generates a hierarchical lattice that is the $d=1$ linear chain. As detailed in the text, around Eq. (6), Ref. [8] also fails to give the correct exact magnetic recursion relation of the simple $d=1$ Ising model, which can be found in textbooks [10,11].

$$- \beta \mathcal{H}^{(n)} = \sum_{\langle ij \rangle} J_{ij}^{(n)} s_i s_j + \sum_{\langle ij \rangle} (\widetilde{H}_{ij}^{(n)} s_i + \widetilde{H}_{ji}^{(n)} s_j) + \sum_i H_i^{(0)} s_i,$$

$$(5)$$

where $\widetilde{H}_{ij}^{(n)}$ is given by the recursion of Eq. (4) above and, again, $H_i^{(0)} \equiv H_i$ is the initial magnetic field appearing in Eq. (2) above.

III. ERROR REFLECTED IN THE $d=1$ ISING MODEL

Incidentally, the simplest hierarchical lattice is the linear one-dimensional system, hierarchically constructed by the repeated replacement of a single bond by two bonds in sequence, as shown in Fig. 1. The exact magnetic-field recursion relation for the simple one-dimensional Ising model is, as is given in introductory classroom textbooks, e.g., [10,11],

$$H' = H + \frac{1}{2}\ln\left(\frac{\mathcal{Z}_{++}\mathcal{Z}_{+-}}{\mathcal{Z}_{--}\mathcal{Z}_{-+}}\right) = H + \frac{1}{2}\ln\left(\frac{\mathcal{Z}_{++}}{\mathcal{Z}_{--}}\right) \qquad (6)$$

($\mathcal{Z}_{+-} = \mathcal{Z}_{-+}$ in the simple one-dimensional Ising system). Thus, as they stand, Eqs. (13) and (14) in [8] should provide the exact recursion relation for the one-dimensional Ising model in the presence of a field. Comparing with Eq. (6) here, it is seen that Eqs. (13) and (14) in Ref. [8] are in error, for the reason explained in the previous paragraph, missing a factor of two (the number of incoming bonds to a site in $d=1$, namely, the coordination number) in their second term on the right-hand side.

IV. ACTUAL OCCURRENCE OF SUPERPOSITION OF THE RENORMALIZATION CONTRIBUTIONS

In the last paragraph of Sec. II of Ref. [8], it is stated that in fact "superpositions," on a given site, of random fields originating from different renormalized bonds joining that given site do indeed occur, as we expected above, and that an *ad hoc* procedure is used in [8] of taking the arithmetic average of these fields (which are nevertheless "superposed"). Thus, incorrectly, [8] uses

$$H'_{\mu} = H_{\mu} + \frac{1}{q_{\mu}}\sum_{\nu}\left[\frac{1}{4}\ln\left(\frac{\mathcal{Z}_{++}^{\mu\nu}\mathcal{Z}_{+-}^{\mu\nu}}{\mathcal{Z}_{--}^{\mu\nu}\mathcal{Z}_{-+}^{\mu\nu}}\right)\right], \qquad (7)$$

where q_{μ} is the coordination number of site μ (and tends to infinity for sites at the highest levels of the construction hierarchy). Obviously no such *ad hocness* should occur in an

exact renormalization-group transformation of a physically realizable system which has its unique thermodynamics. "Superpositions" of fields are in fact truly superpositions of fields. Therefore, as dictated by the exact renormalization-group transformation of a physically realizable system, the sum of the superposed renormalization contributions to the field [second term on the left-hand sides of Eqs. (13) and (14) in [8]] must be used for an exact renormalization-group transformation:

$$H'_{\mu} = H_{\mu} + \sum_{\nu}\left[\frac{1}{4}\ln\left(\frac{\mathcal{Z}_{++}^{\mu\nu}\mathcal{Z}_{+-}^{\mu\nu}}{\mathcal{Z}_{--}^{\mu\nu}\mathcal{Z}_{-+}^{\mu\nu}}\right)\right] \qquad (8)$$

is the correct equation, instead of Eq. (7) above. We note that the magnetic field in Eq. (8) is the magnetic field as defined in Ref. [8], also seen in the above Eq. (1) of this Comment. Erroneously using the arithmetic average enormously weakens the amplitude of the actual renormalization contribution to the field by a division by the coordination number q_{μ} of the site, thus erroneously favoring the spin-glass phase.

V. ERROR REFLECTED IN THE EQUIVALENCE OF THE RANDOM-FIELD AND UNIFORM-FIELD SPIN GLASS

Accordingly, in Ref. [8], the exact renormalization-group transformation of the hierarchical lattice is not achieved. This is also seen by the fact that the outer curve in Fig. 3(a) and the curve in Fig. 4 in [8] should be identical, but erroneously are very different. The uniform-field spin-glass system in the outer curve of Fig. 3(a), by a gauge transformation of a randomly chosen half of its spins, trivially maps [12], exactly, onto the bimodal random-field spin-glass system in Fig. 4, as stated in [7].

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[11] P. M. Chaikin and T. C. Lubensky, *Principles of Condensed Matter Physics* (Cambridge University Press, Cambridge, 1995), p. 247, Eq. 5.6.28.

[12] By contrast, a gauge transformation on half of the spins on a loose-packed lattice, trivially maps, exactly, the bimodal random-field Ising ferromagnet onto the uniform-field Mattis spin glass.